11 ΣΕΙΡΕΣ

11.1 Σειρές με Σταθερούς Όρους

Αριθμητική πρόοδος

Ακολουθία: $a_1 = a,\ \ a_2 = a + d,\ \ a_3 = a + 2d,\ \ \ldots,\ \ a_n = a + (n-1)d$

Άθροισμα: $s_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2}n(a_1 + a_n)$

Γεωμετρική πρόοδος

Ακολουθία: $a_1 = a,\ \ a_2 = ar,\ \ a_3 = ar^2,\ \ \ldots,\ \ a_n = ar^{n-1}$

Άθροισμα: $s_n = a_1 + a_2 + \cdots + a_n = \dfrac{a(1-r^n)}{1-r}, \qquad r \neq 1$

Με $-1 < r < 1$ και $n \to \infty$ έχουμε το άθροισμα άπειρων όρων

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \qquad r \neq 1$$

Δυνάμεις ακεραίων

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$1 + 3 + 5 + 7 + 9 + \cdots + (2n-1) = n^2$$

$$1 + 8 + 16 + 24 + 32 \cdots + 8(n-1) = (2n-1)^2$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

$$1^k + 2^k + 3^k + \cdots + n^k = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1}, \qquad k, n = 1, 2, \ldots$$

[$B_k(x)$ τα πολυώνυμα Bernoulli, B_k οι αριθμοί του Bernoulli]

Αν $S_k = 1^k + 2^k + 3^k + \cdots + n^k$, όπου k και n θετικοί ακέραιοι, τότε

$$\binom{k+1}{1}S_1 + \binom{k+1}{2}S_2 + \cdots + \binom{k+1}{k}S_k = (n+1)^{k+1} - (n+1)$$

Αντίστροφες δυνάμεις θετικών ακεραίων

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \cdots = \frac{\pi\sqrt{3}}{9} + \frac{1}{3}\ln 2$$

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \cdots = \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}\ln(1+\sqrt{2})}{4}$$

$$\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \cdots = \frac{\pi\sqrt{3}}{9} - \frac{1}{3}\ln 2$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = \zeta(3) = 1.2020569032$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \cdots = \zeta(5) = 1.0369277551$$

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}$$

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots = \zeta(x)$$

[$\zeta(x)$ η συνάρτηση ζήτα του Riemann]

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

$$\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^2}+\cdots=\frac{7\pi^4}{720}$$

$$\frac{1}{1^6}-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\cdots=\frac{31\pi^6}{30240}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots=\frac{\pi^4}{96}$$

$$\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\cdots=\frac{\pi^6}{960}$$

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}$$

$$\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{3\pi^2\sqrt{2}}{16}$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots=\frac{1}{2}$$

$$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\cdots=\frac{3}{4}$$

$$\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\frac{1}{7^2\cdot 9^2}+\cdots=\frac{\pi^2-8}{16}$$

$$\frac{1}{1^2\cdot 2^2\cdot 3^2}+\frac{1}{2^2\cdot 3^2\cdot 4^2}+\frac{1}{3^2\cdot 4^2\cdot 5^2}+\cdots=\frac{4\pi^2-39}{16}$$

$$\frac{1}{a}-\frac{1}{a+d}+\frac{1}{a+2d}-\frac{1}{a+3d}+\cdots=\int_0^1\frac{x^{a-1}dx}{1+x^d}$$

$$\frac{1}{1^{2p}}+\frac{1}{2^{2p}}+\frac{1}{3^{2p}}+\frac{1}{4^{2p}}+\cdots=\frac{2^{2p-1}\pi^{2\pi}B_p}{(2p)!}$$

$$\frac{1}{1^{2p}}+\frac{1}{3^{2p}}+\frac{1}{5^{2p}}+\frac{1}{7^{2p}}+\cdots=\frac{(2^{2p}-1)\pi^{2p}B_p}{2(2p)!}$$

$$\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\cdots=\frac{(2^{2p-1}-1)\pi^{2p}B_p}{(2p)!}$$

$$\frac{1}{1^{2p+1}}-\frac{1}{3^{2p+1}}+\frac{1}{5^{2p+1}}-\frac{1}{7^{2p+1}}+\cdots=\frac{\pi^{2p+1}E_p}{2^{2p+2}(2p)!}$$

[E_p οι αριθμοί του Euler]

Με ημίτονα και συνημίτονα

$$\frac{1}{2}+\cos a+\cos 2a+\cdots+\cos na=\frac{\sin\left(n+\frac{1}{2}\right)a}{2\sin(a/2)}$$

$$\sin a+\sin 2a+\sin 3a+\cdots+\sin na=\frac{\sin\left[\frac{1}{2}(n+1)\right]a\sin\frac{1}{2}na}{\sin(a/2)}$$

$$1+r\cos a+r^2\cos 2a+r^3\cos 3a+\cdots=\frac{1-r\cos a}{1-2r\cos a+r^2},\qquad |r|<1$$

$$r\sin a+r^2\sin 2a+r^3\sin 3a+\cdots=\frac{r\sin a}{1-2r\cos a+r^2},\qquad |r|<1$$

$$1+r\cos a+r^2\cos 2a+\cdots+r^n\cos na=\frac{r^{n+2}\cos na-r^{n+1}\cos(n+1)a-r\cos a+1}{1-2r\cos a+r^2}$$

$$r\sin a+r^2\sin 2a+\cdots r^n\sin na=\frac{r\sin a-r^{n+1}\sin(n+1)a+r^{n+2}\sin na}{1-2r\cos a+r^2}$$

11.2 Σειρές Taylor και Maclaurin

Ορισμοί

Αν η $f(x)$ έχει συνεχείς παραγώγους μέχρι της τάξης n, τότε

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\cdots+\frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}+R_n$$

όπου R_n είναι το *υπόλοιπο* μετά από n όρους και με κατάλληλο ξ μεταξύ a και x γράφεται

$$R_n=\frac{f^{(n)}(\xi)(x-a)^n}{n!}\qquad \text{(τύπος του Lagrange)}$$

$$R_n = \frac{f^{(n)}(\xi)(x-\xi)^{n-1}(x-a)}{(n-1)!} \quad \text{(τύπος του Cauchy)}$$

Αν $\lim_{x\to\infty} R_n = 0$, παίρνουμε τη *σειρά Taylor* ή *ανάπτυγμα Taylor* της $f(x)$ για $x = a$. Αν $a = 0$, η σειρά καλείται συχνά *σειρά Maclaurin*. Η σειρά Taylor (ή Maclaurin) συγκλίνει γενικά για κάθε x ενός διαστήματος, που καλείται *διάστημα σύγκλισης*, και αποκλίνει έξω από αυτό το διάστημα.

Για συναρτήσεις δύο μεταβλητών έχουμε

$$f(x, y) = f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b)$$
$$+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\} + \cdots$$

όπου $f_x(a, b)$, $f_y(a, b)$, … είναι οι μερικές παράγωγοι ως προς $x, y, \ldots$ υπολογισμένες στο σημείο $x = a$, $y = b$.

Αναπτύγματα συναρτήσεων

Διωνυμική σειρά

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$
$$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots + \binom{n}{r}a^{n-r}x^r + \cdots$$

$$(a+x)^2 = a^2 + 2ax + x^2$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad -1 < x < 1$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \qquad -1 < x < 1$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots \qquad -1 < x < 1$$

$$(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots \qquad -1 < x < 1$$

$$(1+x)^{-5} = 1 - 5x + 15x^2 - 35x^3 + 70x^4 - \cdots \qquad -1 < x < 1$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots, \qquad -1 < x \le 1$$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \cdots, \qquad -1 < x \le 1$$

$$(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots, \qquad -1 < x \le 1$$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots, \qquad -1 < x \le 1$$

$$(1+x)^{-1/4} = 1 - \frac{1}{4}x + \frac{5}{4\cdot 8}x^2 - \frac{5\cdot 9}{4\cdot 8\cdot 12}x^3 + \cdots, \qquad -1 < x \le 1$$

$$(1+x)^{1/4} = 1 + \frac{1}{4}x - \frac{3}{4\cdot 8}x^2 + \frac{3\cdot 7}{4\cdot 8\cdot 12}x^3 - \cdots, \qquad -1 < x \le 1$$

$$(1+x)^{-3/2} = 1 - \frac{3}{2}x + \frac{3\cdot 5}{2\cdot 4}x^2 - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x^3 + \cdots, \qquad -1 < x < 1$$

$$(1+x)^{-5/2} = 1 - \frac{5}{2}x + \frac{5\cdot 7}{2\cdot 4}x^2 - \frac{5\cdot 7\cdot 9}{2\cdot 4\cdot 6}x^3 + \cdots, \qquad -1 < x < 1$$

Ρητές συναρτήσεις

$$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + \cdots$$

$$\frac{1+x}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots$$

$$\frac{1+6x+x^2}{(1-x)^3} = 1 + 3^2 x + 5^2 x^2 + 7^2 x^3 + \cdots$$

$$\frac{a+(b-a)x}{(1-x)^2} = a + (a+b)x + (a+2b)x^2 + \cdots$$

Τριγωνομετρικές συναρτήσεις

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$$

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

$$\frac{1}{\sin x}=\frac{1}{x}-2x\left\{\frac{1}{x^2-\pi^2}-\frac{1}{x^2-4\pi^2}+\frac{1}{x^2-9\pi^2}-\cdots\right\}$$

$$\frac{1}{\sin^2 x}=\frac{1}{x^2}+\frac{1}{(x-\pi)^2}+\frac{1}{(x+\pi)^2}+\frac{1}{(x-2\pi)^2}+\frac{1}{(x+2\pi)^2}+\cdots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \qquad -\infty<x<\infty$$

$$\cos x=\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\cdots$$

$$\frac{1}{\cos x}=4\pi\left\{\frac{1}{\pi^2-4x^2}-\frac{3}{9\pi^2-4x^2}+\frac{5}{25\pi^2-4x^2}-\cdots\right\}$$

$$\frac{1}{\cos^2 x}=4\left\{\frac{1}{(\pi-2x)^2}+\frac{1}{(\pi+2x)^2}+\frac{1}{(3\pi-2x)^2}+\frac{1}{(3\pi+2x)^2}+\cdots\right\}$$

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots+\frac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!}+\cdots, \qquad |x|<\frac{\pi}{2}$$

$$\tan x=8x\left\{\frac{1}{\pi^2-4x^2}+\frac{1}{9\pi^2-4x^2}+\frac{1}{25\pi^2-4x^2}+\cdots\right\}$$

$$\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\cdots-\frac{2^{2n}B_n x^{2n-1}}{(2n)!}-\cdots, \qquad 0<|x|<\pi$$

$$\cot x=\frac{1}{x}+2x\left\{\frac{1}{x^2-\pi^2}+\frac{1}{x^2-4\pi^2}+\frac{1}{x^2-9\pi^2}+\cdots\right\}$$

$$\sin^{-1} x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots, \qquad |x|<1$$

$$\cos^{-1} x=\frac{\pi}{2}-\sin^{-1} x=\frac{\pi}{2}-\left(x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\cdots\right), \qquad |x|<1$$

$$\tan^{-1} x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, & |x| < 1 \\ \pm\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, & +\text{αν } x \geq 1, -\text{αν } x \leq -1 \end{cases}$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x = \begin{cases} \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right), & |x| < 1 \\ p\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots, & p = 0 \text{ αν } x < 1, p = 1 \text{ αν } x < -1 \end{cases}$$

Εκθετικές και λογαριθμικές συναρτήσεις

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad -\infty < x < \infty$$

$$a^x = e^{x \ln a} = 1 + x \ln a + \frac{(x \ln a)^3}{3!} + \frac{(x \ln a)^3}{3!} + \cdots, \qquad -\infty < x < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \leq 1$$

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots, \qquad -1 < x < 1$$

$$\ln x = \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad x \geq \frac{1}{2}$$

$$\ln x = 2\left\{\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right\}, \qquad x > 0$$

Υπερβολικές συναρτήσεις

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots, \qquad -\infty < x < \infty$$

$$\sinh x = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\cdots$$

$$\frac{1}{\sinh x} = \frac{1}{x} - 2x\left\{\frac{1}{x^2 + \pi^2} + \frac{1}{x^2 + 4\pi^2} + \frac{1}{x^2 + 9\pi^2} + \cdots\right\}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots, \qquad -\infty < x < \infty$$

$$\cosh x = x\left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{9\pi^2}\right)\left(1 + \frac{4x^2}{25\pi^2}\right)\cdots$$

$$\frac{1}{\cosh x} = 4x\left\{\frac{1}{\pi^2 + 4x^2} + \frac{3}{9\pi^2 + 4x^2} + \frac{5}{25\pi^2 + 4x^2} + \cdots\right\}$$

$$\tanh x = x - \frac{x^3}{3} - \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_n x^{2n-1}}{(2n)!} + \cdots, \qquad |x| < \frac{\pi}{2}$$

$$\tanh x = 8x\left\{\frac{1}{\pi^2 + 4x^2} + \frac{1}{9\pi^2 + 4x^2} + \frac{1}{25\pi^2 + 4x^2} + \cdots\right\}$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots \frac{(-1)^{n-1} 2^{2n} B_n x^{2n-1}}{(2n)!} + \cdots, \qquad 0 < |x| < \pi$$

$$\coth x = \frac{1}{x} + 2x\left\{\frac{1}{x^2 + \pi^2} + \frac{1}{x^2 + 4\pi^2} + \frac{1}{x^2 + 9\pi^2} + \cdots\right\}$$

$$\sinh^{-1} x = \begin{cases} x - \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} - \dfrac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots, & |x| < 1 \\ \pm\left(\ln|2x| + \dfrac{1}{2 \cdot 2x^2} + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 4x^4} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6x^6} - \cdots\right), & \begin{array}{l} + \text{αν } x \geq 1, \\ - \text{αν } x \leq -1 \end{array} \end{cases}$$

$$\cosh^{-1} x = \pm\left\{\ln(2x) - \left(\frac{1}{2 \cdot 2x^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 4x^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6x^6} + \cdots\right)\right\}, \qquad \begin{array}{c} + \text{αν } \cosh x > 0 \\ - \text{αν } \cosh x < 0 \\ (x \geq 1) \end{array}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots, \qquad |x| < 1$$

$$\coth^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots, \qquad |x| > 1$$

Διάφορες σειρές

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \cdots, \qquad -\infty < x < \infty$$

$$e^{\cos x} = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + \cdots\right), \qquad -\infty < x < \infty$$

$$e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \cdots, \qquad |x| < \frac{\pi}{2}$$

$$e^x \sin x = x + x^2 + \frac{2x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \cdots + \frac{2^{n/2}\sin(n\pi/4)x^n}{n!} + \cdots, \qquad -\infty < x < \infty$$

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots + \frac{2^{n/2}\cos(n\pi/4)x^n}{n!} + \cdots, \qquad -\infty < x < \infty$$

$$\ln|\sin x| = \ln|x| - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots - \frac{2^{2n-1}B_n x^{2n}}{n(2n)!} + \cdots, \qquad 0 < |x| < \pi$$

$$\ln|\cos x| = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots - \frac{2^{2n-1}(2^{2n}-1)B_n x^{2n}}{n(2n)!} + \cdots, \qquad |x| < \frac{\pi}{2}$$

$$\ln|\tan x| = \ln|x| + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots + \frac{2^{2n}(2^{2n-1}-1)B_n x^{2n}}{n(2n)!} + \cdots, \qquad 0 < |x| < \frac{\pi}{2}$$

$$\frac{\ln(1+x)}{1+x} = x - \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 - \cdots, \qquad |x|$$

Αντίστροφη σειρά

Αν $\quad y = c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots$

τότε $\quad x = C_1y + C_2y^2 + C_3y^3 + C_4y^4 + C_5y^5 + C_6y^6 + \cdots$

όπου

$c_1C_1 = 1$

$c_1^3C_2 = -c_2$

$c_1^5C_3 = 2c_2^2 - c_1c_3$

$c_1^7C_4 = 5c_1c_2c_3 - 5c_2^3 - c_1^2c_4$

$c_1^9C_5 = 6c_1^2c_2c_4 + 3c_1^2c_3^2 - c_1^3c_5 + 14c_2^4 - 21c_1c_2^2c_3$

$c_1^{11}C_6 = 7c_1^3c_2c_5 + 84c_1c_2^3c_3 + 7c_1^3c_3c_4 - 28c_1^2c_2c_3^2 - c_1^4c_6 - 28c_1^2c_2^2c_4 - 42c_2^5$