20 ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΣ FOURIER

20.1 Το Ολοκληρωτικό Θεώρημα του Fourier

Αν (i) οι συναρτήσεις $f(t)$ και $f'(t)$ είναι τμηματικά συνεχείς σε κάθε πεπερασμένο διάστημα $-L < t < L$, (ii) το $\int_{-\infty}^{\infty} |f(t)|\,dt$ συγκλίνει και (iii) η $f(t)$ ισούται με $\frac{1}{2}\{f(t+0)+f(t-0)\}$ σε κάθε σημείο ασυνέχειας, τότε

$$f(t) = \int_0^{\infty} [A(\omega)\cos\omega t + B(\omega)\sin\omega t]\,d\omega$$

όπου

$$\begin{cases} A(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(u)\cos\omega u\,du \\[2ex] B(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(u)\sin\omega u\,du \end{cases}$$

Δύο άλλες μορφές του *θεωρήματος του Fourier* είναι η

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\omega(u-t)\,du\,d\omega$$

και η

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\left[\int_{-\infty}^{\infty} f(u)e^{i\omega u}\,du\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\omega(u-t)}\,du\,d\omega$$

Αν η $f(t)$ είναι περιττή συνάρτηση [δηλ. $f(-t) = -f(t)$], τότε

$$f(t) = \frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(u)\sin\omega u\,du\right]\sin\omega t\,d\omega$$

Αν η $f(t)$ είναι άρτια συνάρτηση [δηλ. $f(-t) = f(t)$], τότε

$$f(t) = \frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(u)\cos\omega u\,du\right]\cos\omega t\,d\omega$$

Ουσιαστικά, όλα τα προηγούμενα συνοψίζονται στο εξής: Όλη η πληροφορία που υπάρχει στο χώρο t [δηλ. στην $f(t)$] μπορεί με μια ολοκλήρωση (ως προς u) να μεταφερθεί στο χώρο ω και μετά, με μια δεύτερη ολοκλήρωση, πίσω στο χώρο t.

20.2 Μετασχηματισμένες Fourier

Η *μετασχηματισμένη Fourier* της $f(t)$ ορίζεται με τη σχέση

$$F(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Η *αντίστροφη μετασχηματισμένη Fourier* της $F(\omega)$ ορίζεται με τη σχέση

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$

Οι $f(t)$ και $F(\omega)$ καλούνται *ζεύγος μετασχηματισμένων Fourier.* Η $f(t)$ αντιπροσωπεύει την πληροφορία στο χώρο του χρόνου και η $F(\omega)$ στο χώρο των συχνοτήτων (συνήθως το t παριστάνει χρόνο και το ω συχνότητα).

Ιδιότητες

Αν $F(\omega) = \mathcal{F}\{f(t)\}$ και $G(\omega) = \mathcal{F}\{g(t)\}$, τότε με a και b σταθερές έχουμε

$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$	Γραμμικότητα
$\mathcal{F}\{f(at)\} = a^{-1}F(\omega/a)$	Αλλαγή κλίμακας
$\mathcal{F}\{f(t+a)\} = e^{-ia\omega}F(\omega)$	Μετατόπιση
$\mathcal{F}\{t^n f(t)\} = (-i)^n \dfrac{d^n F}{d\omega^n}$	Πολλαπλασιασμός επί δύναμη
$\mathcal{F}\{f(t)e^{iat}\} = F(\omega - a)$	Πολλαπλασιασμός επί e^{iat}

Αν $\lim_{a\to 0} \mathcal{F}\{f(t,a)\} = \mathcal{F}\{f(t)\}$, τότε $\lim_{a\to 0} f(t,a)\} = f(t)$ όπου η $f(t)$ είναι συνεχής.

Αν επιπλέον (α) υπάρχουν οι παράγωγοι $f^{(r)}(t)$ μέχρι και τάξης n της $f(t)$ για κάθε t και (β) $f^{(r)}(t) \to 0$ για $|t| \to \infty$ και κάθε $r < n$, τότε

$$\mathcal{F}\{f^{(n)}(t)\} = (-i\omega)^n F(\omega)$$

Το θεώρημα της συνέλιξης

Αν $f * g = \int_{-\infty}^{\infty} f(u) g(t-u) du$ είναι η *συνέλιξη* δύο συναρτήσεων $f(t)$ και $g(t)$, τότε

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$$

δηλ. η μετασχηματισμένη της συνέλιξης ισούται με το γινόμενο των μετασχηματισμένων. Η σχέση αυτή χρησιμοποιείται και στη μορφή $f * g = \mathcal{F}^{-1}[F(\omega)G(\omega)]$.

Ταυτότητα του Parseval

Αν $F(\omega) = \mathcal{F}\{f(t)\}$ και $G(\omega) = \mathcal{F}\{g(t)\}$, τότε

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega$$

όπου αστερίσκος σαν πάνω δείκτης σημαίνει το συζυγή μιγαδικό. Ειδικότερα

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

20.3 Ημιτονοειδής και Συνημιτονοειδής Μετασχηματισμός Fourier

Η *ημιτονοειδής μετασχηματισμένη Fourier* της $f(t)$ είναι

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\sin\omega t\,dt$$

Η *αντίστροφη ημιτονοειδής μετασχηματισμένη Fourier* της $F_s(\omega)$ είναι

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega)\sin\omega t\,dt$$

Η *συνημιτονοειδής μετασχηματισμένη Fourier* της $f(t)$ είναι

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\cos\omega t\,dt$$

Η *αντίστροφη συνημιτονοειδής μετασχηματισμένη Fourier* της $F_c(\omega)$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega)\cos\omega t\,d\omega$$

20.4 Πίνακας Μετασχηματισμένων Fourier

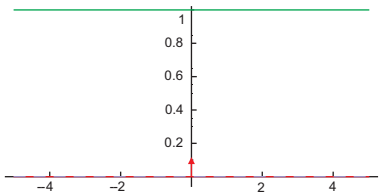
Στα επόμενα δίνονται σε κάθε περίπτωση (α) η (πραγματική) συνάρτηση $f(t)$ και (β) η αντίστοιχη μετασχηματισμένη Fourier $F(\omega)$ [ή $F_s(\omega)$ ή $F_c(\omega)$]. Επίσης δίνονται οι γραφικές παραστάσεις (γ) της $f(t)$ με πράσινο, (δ) της $\mathrm{Re}\{F(\omega)\}$ με κόκκινο, (ε) της $\mathrm{Im}\{F(\omega)\}$ με μωβ. Στον οριζόντιο άξονα οι αριθμοί είναι τιμές των t και ω συγχρόνως. Στον κατακόρυφο άξονα οι αριθμοί είναι τιμές των $f(t)$ και $F(\omega)$ συγχρόνως. Έτσι φαίνονται η μορφή της $f(t)$ και η κατανομή της $F(\omega)$.

Για μερικές $f(t)$ το $\int_{-\infty}^{\infty} |f(t)|\,dt$ δεν υπάρχει, αλλά η $F(\omega)$ μπορεί να χρησιμοποιηθεί σε τυπικούς (όχι αυστηρούς) υπολογισμούς.

Μετασχηματισμένες Fourier ($f(t)$, $\mathrm{Re}\{F(\omega)\}$, $\mathrm{Im}\{F(\omega)\}$, $-\infty < \omega < \infty$)

$$f(t) = 1$$

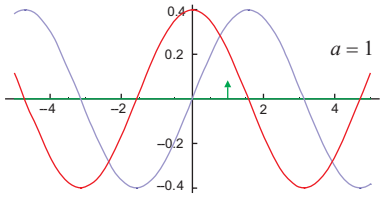
$$F(\omega) = \sqrt{2\pi}\,\delta(\omega)$$



Σχ. 20-1

$$f(t) = \delta(t-a)$$

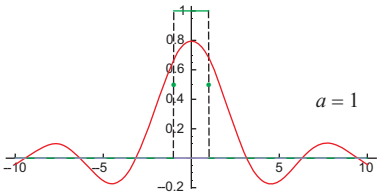
$$F(\omega) = \frac{1}{\sqrt{2\pi}}\,e^{ia\omega}$$



Σχ. 20-2

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases} \qquad a > 0$$

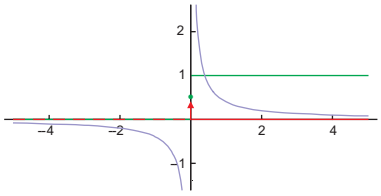
$$F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin a\omega}{\omega}$$



Σχ. 20-3

$$f(t) = \tfrac{1}{2}[1 + \mathrm{sign}(t)]$$

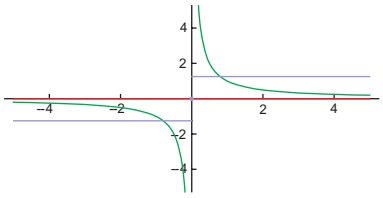
$$F(\omega) = \sqrt{\frac{\pi}{2}}\,\delta(\omega) + \frac{i}{\sqrt{2\pi}\,\omega}$$



Σχ. 20-4

$$f(t) = \frac{1}{t}$$

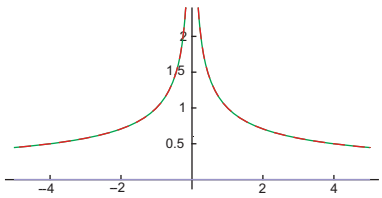
$$F(\omega) = i\sqrt{\frac{\pi}{2}}\,\mathrm{sign}(\omega)$$



Σχ. 20-5

$$f(t) = \frac{1}{\sqrt{|t|}}$$

$$F(\omega) = \frac{1}{\sqrt{|\omega|}}$$

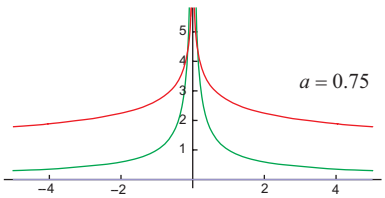


Σχ. 20-6

$$f(t)=\frac{1}{|t|^a},\qquad 0<a<1$$

$$F(\omega)=\sqrt{\frac{2}{\pi}}\,|\,\omega\,|^{a-1}\,\Gamma(1-a)\sin\left(\frac{a\pi}{2}\right)$$

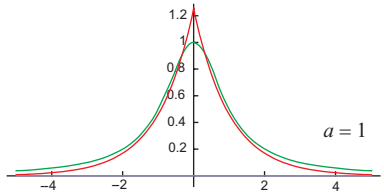
Σχ. 20-7



$$f(t)=\frac{1}{t^2+a^2},\qquad a>0$$

$$F(\omega)=\sqrt{\frac{\pi}{2}}\,\frac{e^{-a|\omega|}}{a}$$

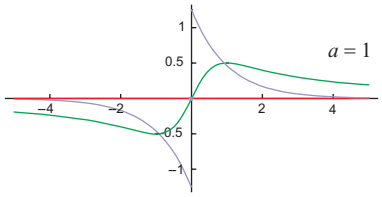
Σχ. 20-8



$$f(t)=\frac{t}{t^2+a^2},\qquad a>0$$

$$F(\omega)=i\sqrt{\frac{\pi}{2}}\,\mathrm{sign}(\omega)\,e^{-a|\omega|}$$

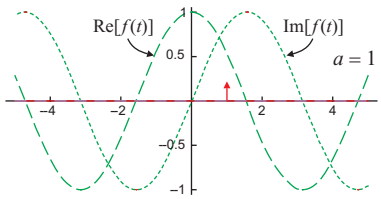
Σχ. 20-9



$$f(t)=e^{iat}$$

$$F(\omega)=\sqrt{2\pi}\,\delta(\omega+a)$$

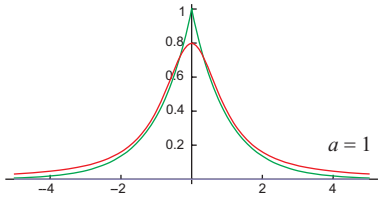
Σχ. 20-10



$$f(t)=e^{-a|t|},\qquad a>0$$

$$F(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{a}{\omega^2+a^2}$$

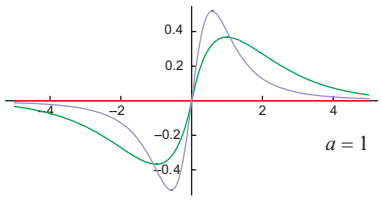
Σχ. 20-11



$$f(t)=t\,e^{-a|t|},\qquad a>0$$

$$F(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{2a\omega i}{(\omega^2+a^2)^2}$$

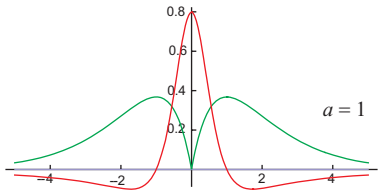
Σχ. 20-12



$$f(t) = |t|e^{-a|t|}, \qquad a > 0$$

$$F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{a^2-\omega^2}{(\omega^2+a^2)^2}$$

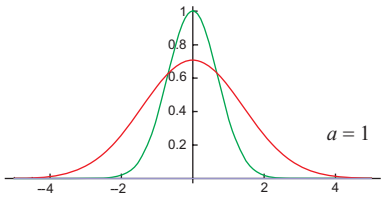
Σχ. 20-13



$$f(t) = \exp(-at^2), \qquad a > 0$$

$$F(\omega) = \frac{1}{\sqrt{2a}}\exp\left(-\frac{\omega^2}{4a}\right)$$

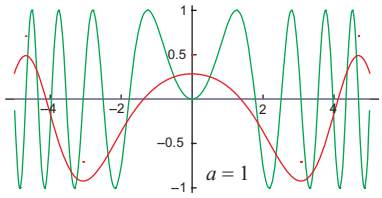
Σχ. 20-14



$$f(t) = \sin(at^2), \qquad a > 0$$

$$F(\omega) = \frac{1}{\sqrt{2a}}\cos\left(\frac{\omega^2}{4a}+\frac{\pi}{4}\right)$$

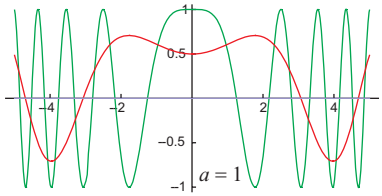
Σχ. 20-15



$$f(t) = \cos(at^2)$$

$$F(\omega) = \frac{1}{\sqrt{2|a|}}\cos\left(\frac{\omega^2}{4|a|}-\frac{\pi}{4}\right)$$

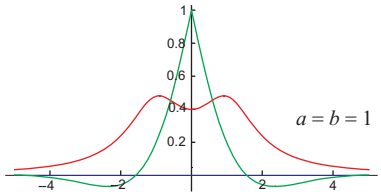
Σχ. 20-16



$$f(t) = e^{-a|t|}\cos bt, \qquad a > 0$$

$$F(\omega) = \frac{a}{\sqrt{2\pi}}\left[\frac{1}{(\omega+b)^2+a^2}+\frac{1}{(\omega-b)^2+a^2}\right]$$

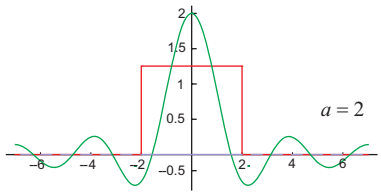
Σχ. 20-17



$$f(t) = t^{-1}\sin at, \qquad a > 0$$

$$F(\omega) = \frac{1}{2}\sqrt{\frac{\pi}{2}}[1+\operatorname{sign}(a-|\omega|)]$$

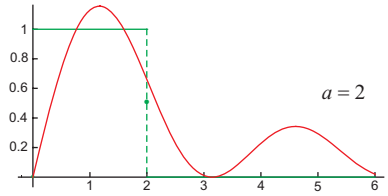
Σχ. 20-18



Ημιτονοειδείς Μετασχηματισμένες Fourier ($\text{—}\, f(t)$, $\text{—}\, F_s(\omega)$, $0 < \omega < \infty$)

$$f(t) = \begin{cases} 1, & 0 < t < a, \\ 0, & t > a \end{cases} \qquad a > 0$$

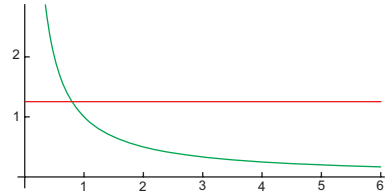
$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos a\omega}{\omega}$$



Σχ. 20-19

$$f(t) = t^{-1}$$

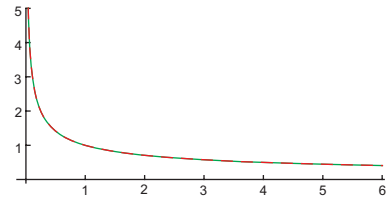
$$F_s(\omega) = \sqrt{\frac{\pi}{2}}$$



Σχ. 20-20

$$f(t) = \frac{1}{\sqrt{t}}$$

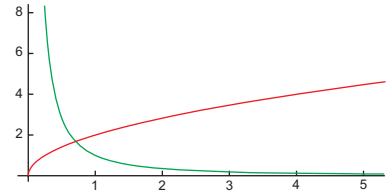
$$F_s(\omega) = \frac{1}{\sqrt{\omega}}$$



Σχ. 20-21

$$f(t) = t^{-3/2}$$

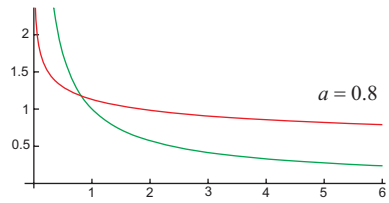
$$F_s(\omega) = 2\sqrt{\omega}$$



Σχ. 20-22

$$f(t) = t^{-a}, \qquad 0 < a < 2$$

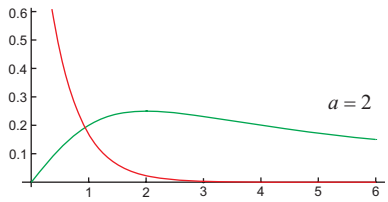
$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\,\Gamma(1-a)\,\omega^{a-1}\cos\frac{a\pi}{2}$$



Σχ. 20-23

$$f(t) = \frac{t}{t^2 + a^2}, \qquad a > 0$$

$$F_s(\omega) = \sqrt{\frac{\pi}{2}}\,e^{-a\omega}$$

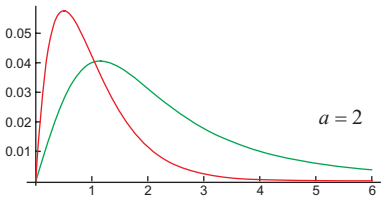


Σχ. 20-24

$$f(t)=\frac{t}{(t^2+a^2)^2}, \qquad a>0$$

$$F_s(\omega)=\sqrt{\frac{\pi}{2}}\,\frac{\omega}{2a}\,e^{-a\omega}$$

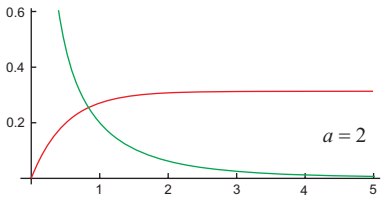
Σχ. 20-25



$$f(t)=\frac{1}{t(t^2+a^2)}, \qquad a>0$$

$$F_s(\omega)=\sqrt{\frac{\pi}{2}}\,\frac{1-e^{-a\omega}}{a^2}$$

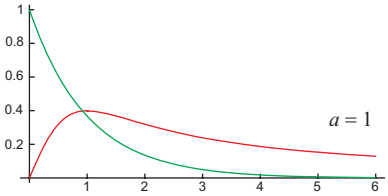
Σχ. 20-26



$$f(t)=e^{-at}, \qquad a>0$$

$$F_s(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{\omega}{\omega^2+a^2}$$

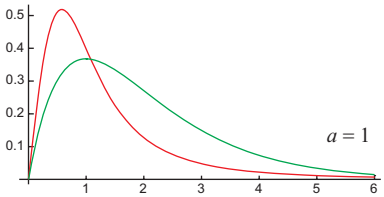
Σχ. 20-27



$$f(t)=te^{-at}, \qquad a>0$$

$$F_s(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{2a\omega}{(\omega^2+a^2)^2}$$

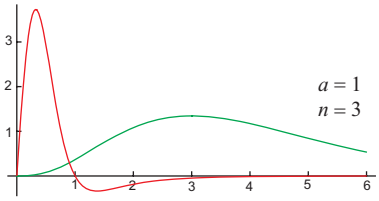
Σχ. 20-28



$$f(t)=t^n e^{-at}, \qquad a>0,\ n>-2$$

$$F_s(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n+1)\sin[(n+1)\tan^{-1}(\omega/a)]}{(\omega^2+a^2)^{(n+1)/2}}$$

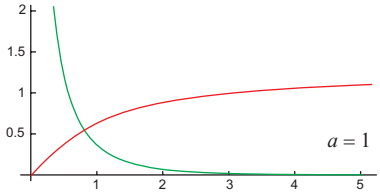
Σχ. 20-29



$$f(t)=\frac{1}{t}e^{-at}, \qquad a>0$$

$$F_s(\omega)=\sqrt{\frac{2}{\pi}}\,\tan^{-1}\left(\frac{\omega}{a}\right)$$

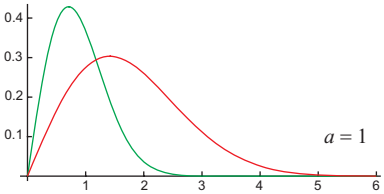
Σχ. 20-30



$$f(t) = t\exp(-at^2), \qquad a > 0$$

$$F_S(\omega) = \frac{\omega}{(2a)^{3/2}} \exp\left(\frac{-\omega^2}{4a}\right)$$

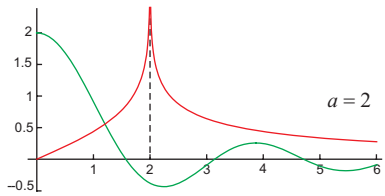
Σχ. 20-31



$$f(t) = \frac{\sin at}{t}, \qquad a > 0$$

$$F_s(\omega) = \frac{1}{\sqrt{2\pi}} \ln\left|\frac{\omega + a}{\omega - a}\right|$$

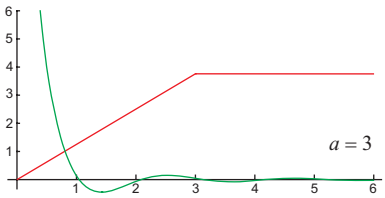
Σχ. 20-32



$$f(t) = \frac{\sin at}{t^2}, \qquad a > 0$$

$$F_s(\omega) = \begin{cases} \sqrt{\pi/2}\,\omega, & \omega \le a \\ \sqrt{\pi/2}\,a, & \omega > a \end{cases}$$

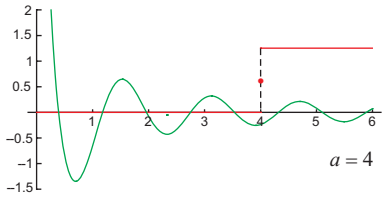
Σχ. 20-33



$$f(t) = \frac{\cos at}{t}$$

$$F_s(\omega) = \begin{cases} 0, & \omega < a \\ \sqrt{\pi/8}, & \omega = a \\ \sqrt{\pi/2}, & \omega > a \end{cases}$$

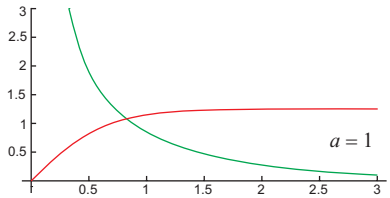
Σχ. 20-34



$$f(t) = \frac{1}{\sinh at}$$

$$F_s(\omega) = \sqrt{\frac{\pi}{2}}\,\frac{1}{a} \tanh\frac{\pi\omega}{2a}$$

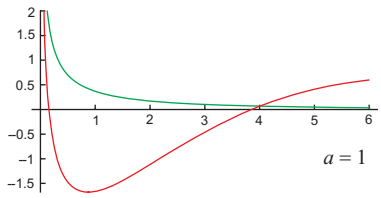
Σχ. 20-35



$$f(t) = \tan^{-1}\left(\frac{t}{a}\right), \qquad a > 0$$

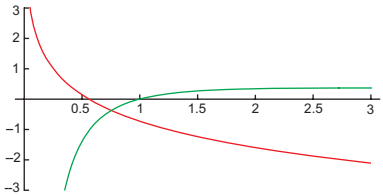
$$F_s(\omega) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a\omega}}{\omega}$$

Σχ. 20-36



$$f(t)=\frac{\ln t}{t}$$

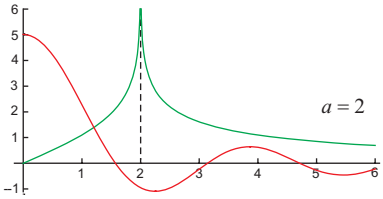
$$F_s(\omega)=-\sqrt{\frac{\pi}{2}}(\gamma+\ln\omega)$$



Σχ. 20-37

$$f(t)=\ln\left|\frac{t+a}{t-a}\right|,\qquad a>0$$

$$F_s(\omega)=\sqrt{2\pi}\,\frac{\sin a\omega}{\omega}$$



Σχ. 20-38

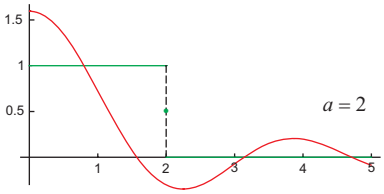
$$f(x)=\frac{1}{e^{2x}-1}$$

$$F_S(\omega)=\frac{\pi}{4}\coth\left(\frac{\pi\omega}{2}\right)-\frac{1}{2\omega}$$

Συνημιτονοειδείς Μετασχηματισμένες Fourier (— $f(t)$, — $F_c(\omega)$, $0<\omega<\infty$)

$$f(t)=\begin{cases}1, & 0<t<a,\\ 0, & t>a\end{cases}\qquad a>0$$

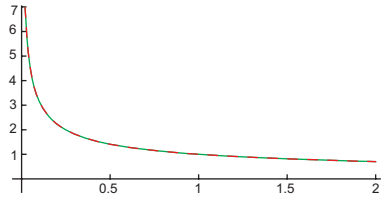
$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{\sin a\omega}{\omega}$$



Σχ. 20-39

$$f(t)=\frac{1}{\sqrt{t}}$$

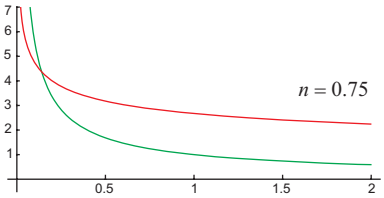
$$F_c(\omega)=\frac{1}{\sqrt{\omega}}$$



Σχ. 20-40

$$f(t)=t^{-n},\qquad 0<n<1$$

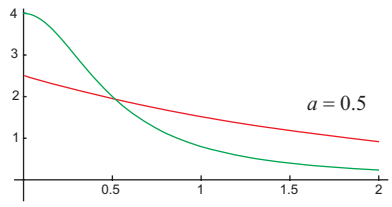
$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\,\omega^{n-1}\Gamma(1-n)\sin(n\pi/2)$$



Σχ. 20-41

$$f(t)=\frac{1}{t^2+a^2}, \qquad a>0$$

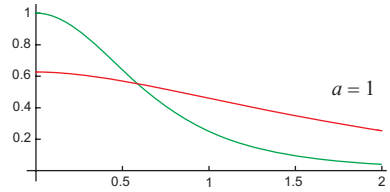
$$F_c(\omega)=\sqrt{\frac{\pi}{2}}\,\frac{e^{-a\omega}}{a}$$



Σχ. 20-42

$$f(t)=\frac{1}{(t^2+a^2)^2}, \qquad a>0$$

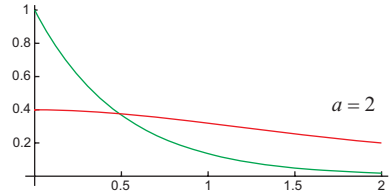
$$F_c(\omega)=\sqrt{\frac{\pi}{2}}\,\frac{(1+a\omega)e^{-a\omega}}{2a^3}$$



Σχ. 20-43

$$f(t)=e^{-at}, \qquad a>0$$

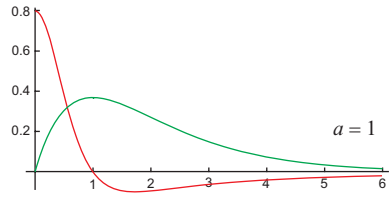
$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{a}{\omega^2+a^2}$$



Σχ. 20-44

$$f(t)=te^{-at}, \qquad a>0$$

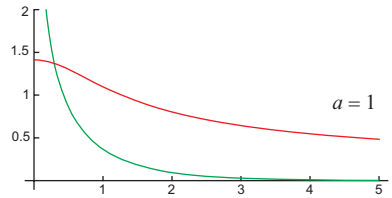
$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{a^2-\omega^2}{(\omega^2+a^2)^2}$$



Σχ. 20-45

$$f(t)=\frac{e^{-at}}{\sqrt{t}}, \qquad a>0$$

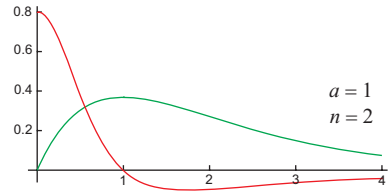
$$F_c(\omega)=\sqrt{\frac{a+\sqrt{\omega^2+a^2}}{\omega^2+a^2}}$$



Σχ. 20-46

$$f(t)=t^{n-1}e^{-at}, \qquad a>0,\ n>0$$

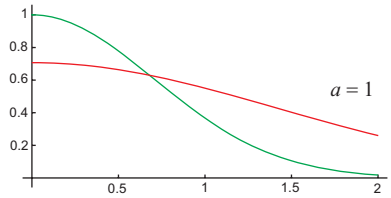
$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\,\Gamma(n)\,\frac{\cos[n\tan^{-1}(\omega/a)]}{(\omega^2+a^2)^{n/2}}$$



Σχ. 20-47

$$f(t)=\exp\left(-at^2\right),\qquad a>0$$

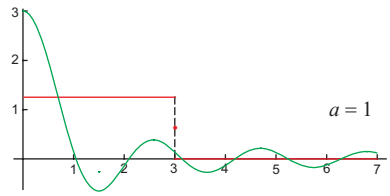
$$F_c(\omega)=\frac{1}{\sqrt{2a}}\exp\left(\frac{-\omega^2}{4a}\right)$$



Σχ. 20-48

$$f(t)=\frac{\sin at}{t},\qquad a>0$$

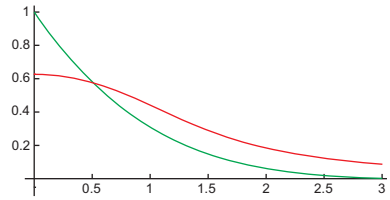
$$F_c(\omega)=\frac{1}{2}\sqrt{\frac{\pi}{2}}\,[1+\operatorname{sign}(a-\omega)]$$



Σχ. 20-49

$$f(t)=t^{-1}e^{-t}\sin t$$

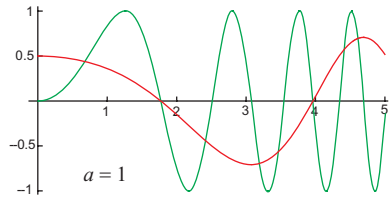
$$F_c(\omega)=\frac{1}{\sqrt{2\pi}}\tan^{-1}\left(\frac{2}{\omega^2}\right)$$



Σχ. 20-50

$$f(t)=\sin(at^2),\qquad a>0$$

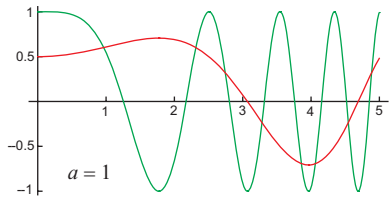
$$F_c(\omega)=\frac{1}{2\sqrt{a}}\left(\cos\frac{\omega^2}{4a}-\sin\frac{\omega^2}{4a}\right)$$



Σχ. 20-51

$$f(t)=\cos(at^2),\qquad a>0$$

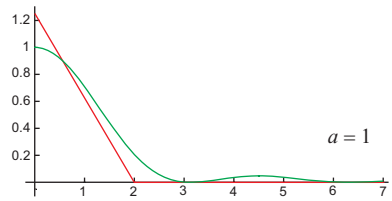
$$F_c(\omega)=\frac{1}{2\sqrt{a}}\left(\cos\frac{\omega^2}{4a}+\sin\frac{\omega^2}{4a}\right)$$



Σχ. 20-52

$$f(t)=t^{-2}\sin^2 at,\qquad a>0$$

$$F_c(\omega)=\begin{cases}\sqrt{\frac{\pi}{2}}\left(a-\frac{1}{2}\omega\right), & \omega<2a\\ 0, & \omega\geq 2a\end{cases}$$

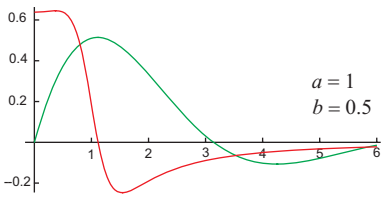


Σχ. 20-53

$$f(t) = e^{-bt}\sin at, \qquad a > 0,\ b > 0$$

$$F_c(\omega) = \frac{1}{\sqrt{2\pi}}\left[\frac{a+\omega}{b^2+(a+\omega)^2} + \frac{a-\omega}{b^2+(a-\omega)^2}\right]$$

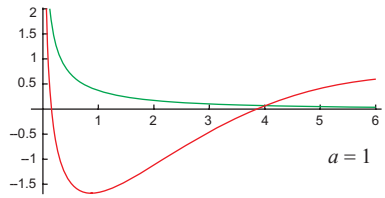
Σχ. 20-54



$$f(x) = \frac{e^{-a\sqrt{x}}}{\sqrt{x}}$$

$$F_c(\omega) = \sqrt{\frac{\pi}{2\omega}}\left\{\cos(2a\sqrt{\omega}) - \sin(2a\sqrt{\omega})\right\}$$

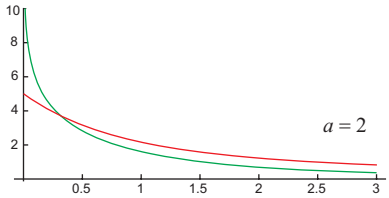
Σχ. 20-55



$$f(t) = \ln\left(1 + \frac{a^2}{t^2}\right), \qquad a > 0$$

$$F_c(\omega) = \sqrt{2\pi}\,\frac{1 - e^{-a\omega}}{\omega}$$

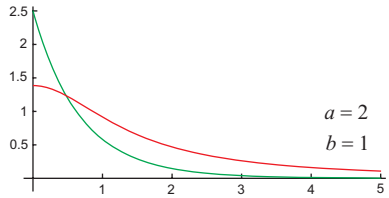
Σχ. 20-56



$$f(t) = \ln\left(\frac{a^2+t^2}{b^2+t^2}\right), \qquad a > 0,\ b > 0$$

$$F_c(\omega) = \sqrt{2\pi}\,\frac{e^{-b\omega} - e^{-a\omega}}{\omega}$$

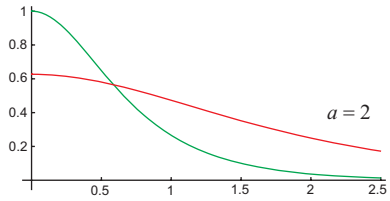
Σχ. 20-57



$$f(t) = \frac{1}{\cosh at}, \qquad a > 0$$

$$F_c(\omega) = \sqrt{\frac{\pi}{2}}\,\frac{1}{a\cosh(\pi\omega/2a)}$$

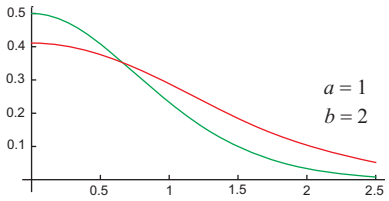
Σχ. 20-58



$$f(t) = \frac{\sinh at}{\sinh bt}, \qquad 0 < a < b$$

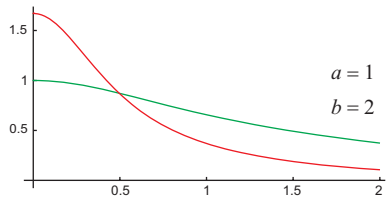
$$F_c(\omega) = \sqrt{\frac{\pi}{2}}\,\frac{\sinh(\pi a/b)}{b[\cos(\pi\omega/b) + \cos(\pi a/b)]}$$

Σχ. 20-59



$$f(t) = \frac{\cosh at}{\cosh bt}, \qquad 0 < a < b$$

$$F_c(\omega) = \sqrt{2\pi}\,\frac{\cos[\pi a/(2b)]\cosh[\pi\omega/(2b)]}{b[\cos(\pi\omega/b) + \cos(\pi a/b)}$$



Σχ. 20-60